

MAXIMAL FUNCTION AND CARLESON MEASURES IN BÉKOLLÉ-BONAMI WEIGHTS

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ABSTRACT. Let ω be a Békollé-Bonami weight. We give a complete characterization of the positive measures μ such that

$$\int_{\mathcal{H}} |M_{\omega} f(z)|^q d\mu(z) \leq C \left(\int_{\mathcal{H}} |f(z)|^p \omega(z) dV(z) \right)^{q/p}$$

and

$$\mu(\{z \in \mathcal{H} : Mf(z) > \lambda\}) \leq \frac{C}{\lambda^q} \left(\int_{\mathcal{H}} |f(z)|^p \omega(z) dV(z) \right)^{q/p}$$

where M_{ω} is the weighted Hardy-Littlewood maximal function on the upper-half plane \mathcal{H} , and $1 \leq p, q < \infty$.

1. INTRODUCTION

Let \mathcal{H} be the upper-half plane, that is the set $\{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, \text{ and } y > 0\}$. Given ω a nonnegative locally integrable function on \mathcal{H} (i.e a weight), and $1 \leq p < \infty$, we denote by $L_{\omega}^p(\mathcal{H})$, the set of functions f defined on \mathcal{H} such that

$$\|f\|_{p,\omega}^p := \int_{\mathbb{D}} |f(z)|^p \omega(z) dV(z) < \infty$$

with dV being the Lebesgue measure on \mathcal{H} .

Given a weight ω , and $1 < p < \infty$, we say ω is in the Békollé-Bonami class B_p , if

$$[\omega]_{B_p} := \sup_{I \subset \mathbb{R}, I \text{ interval}} \left(\frac{1}{|Q_I|} \int_{Q_I} \omega(z) dV(z) \right) \left(\frac{1}{|Q_I|} \int_{Q_I} \omega(z)^{1-p'} dV(z) \right)^{p-1} < \infty,$$

$Q_I := \{z = x + iy \in \mathbb{C} : x \in I \text{ and } 0 < y < |I|\}$, $|Q_I| = \int_{Q_I} dV(z)$, $pp' = p + p'$. This is the exact range of weights ω for which the orthogonal projection P from $L^2(\mathcal{H}, dV(z))$ to its closed subspace consisting of analytic functions is bounded on $L_{\omega}^p(\mathcal{H})$ (see [2, 3, 18, 20]).

Let $1 < p < \infty$, and $\omega \in B_p$. We provide in this note a full characterization of positive measures μ on \mathcal{H} such that the following Carleson-type embedding

$$(1) \quad \int_{\mathcal{H}} |M_{\omega} f(z)|^q d\mu(z) \leq C \left(\int_{\mathcal{H}} |f(z)|^p \omega(z) dV(z) \right)^{q/p}$$

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holds when $p \leq q < \infty$ and when $p > q$, where M_ω is the weighted Hardy-Littlewood maximal function,

$$M_\omega f(z) := \sup_{I \text{ interval in } \mathbb{R}, z \in Q_I} \frac{1}{|Q_I|_\omega} \int_{Q_I} |f(z)| \omega(z) dV(z),$$

$$|Q_I|_\omega = \omega(Q_I) = \int_{Q_I} \omega(z) dV(z).$$

We also characterize those positive measures μ on \mathcal{H} such that

$$(2) \quad \mu(\{z \in \mathcal{H} : Mf(z) > \lambda\}) \leq \frac{C}{\lambda^q} \left(\int_{\mathcal{H}} |f(z)|^p \omega(z) dV(z) \right)^{q/p}$$

where M is the unweighted Hardy-Littlewood maximal function ($M = M_\omega$ with $\omega(z) = 1$ for all $z \in \mathcal{H}$).

Before stating our main results, let us see how the above questions are related to some others in complex analysis. We recall that the Bergman space $A_\omega^p(\mathcal{H})$ is the subspace of $L_\omega^p(\mathcal{H})$ consisting of holomorphic functions on \mathcal{H} . The usual Bergman spaces in the unit disc of \mathbb{C} or the unit ball of \mathbb{C}^n correspond to the weights $\omega(z) = (1 - |z|^2)^\alpha dA(z)$, $\alpha > -1$. A positive measure on \mathcal{H} is called a q -Carleson measure for $A_\omega^p(\mathcal{H})$ if there is a constant $C > 0$ such that for any $f \in A_\omega^p(\mathcal{H})$,

$$(3) \quad \int_{\mathcal{H}} |f(z)|^q d\mu(z) \leq C \left(\int_{\mathcal{H}} |f(z)|^p \omega(z) dV(z) \right)^{q/p}.$$

Carleson measures are very useful in the study of many other questions in complex and harmonic analysis: Toeplitz operators, Cesàro-type integrals, embeddings between different analytic function spaces, etc... Carleson measures for Bergman spaces with standard weights $\omega(z) = (1 - |z|^2)^\alpha dA(z)$ in the unit disc and the unit ball of \mathbb{C}^n , $\alpha > -1$ have been studied in [4, 10, 15, 16, 17, 21]. The case of Bergman spaces of the unit disc of \mathbb{C} with Békollé-Bonami weights has been handled in [5, 9].

Let us suppose that $\omega \in B_p$. Applying the mean value property one obtains that there is a constant $C > 0$ such that for any $f \in A_\omega^p(\mathcal{H})$, and for any $z \in \mathcal{H}$, if I is the unique interval such that Q_I is centered at z , then

$$|f(z)| \leq \frac{C}{\omega(Q_I)} \int_{Q_I} |f(w)| \omega(w) dV(w).$$

It follows that any measure satisfying (1) is a q -Carleson measure for $A_\omega^p(\mathcal{H})$.

Our first main result is the following.

THEOREM 1.1. *Let $1 < p \leq q < \infty$, and ω a weight on \mathcal{H} . Assume that $\omega \in B_p$. Then the following assertions are equivalent.*

(1) *There exists a constant $C_1 > 0$ such that for any $f \in L_\omega^p(\mathcal{H})$,*

$$(4) \quad \left(\int_{\mathcal{H}} |M_\omega f(z)|^q d\mu(z) \right)^{1/q} \leq C_1 \|f\|_{p,\omega}.$$

(2) *There is a constant C_2 such that for any interval $I \subset \mathbb{R}$,*

$$(5) \quad \mu(Q_I) \leq C_2 (\omega(Q_I))^{\frac{q}{p}}.$$

Our next result provides estimations with loss.

THEOREM 1.2. *Let $1 < q < p < \infty$, and ω a weight on \mathcal{H} . Assume that $\omega \in B_p$. Then (4) holds if and only if the function*

$$(6) \quad K_\mu(z) := \sup_{I \subset \mathbb{R}, I \text{ interval}, z \in Q_I} \frac{\mu(Q_I)}{\omega(Q_I)}$$

belongs to $L_\omega^s(\mathcal{H})$ where $s = \frac{p}{p-q}$.

Our last result provides weak-type estimates.

THEOREM 1.3. *Let $1 \leq p, q < \infty$, and ω a weight on \mathcal{H} . Then the following assertions are equivalent.*

- (a) *There is a constant $C_1 > 0$ such that for any $f \in L_\omega^p(\mathcal{H})$, and any $\lambda > 0$,*

$$(7) \quad \mu(\{z \in \mathcal{H} : Mf(z) > \lambda\}) \leq \frac{C_1}{\lambda^q} \left(\int_{\mathcal{H}} |f(z)|^p \omega(z) dV(z) \right)^{q/p}$$

- (b) *There is a constant $C_2 > 0$ such that for any interval $I \subset \mathbb{R}$,*

$$(8) \quad |Q_I|^{-q/p} \left(\frac{1}{|Q_I|} \int_{Q_I} \omega^{1-p'}(z) dV(z) \right)^{q/p'} \mu(Q_I) \leq C_1$$

where $\left(\frac{1}{|Q_I|} \int_{Q_I} \omega^{1-p'}(z) dV(z) \right)^{1/p'}$ is understood as $(\inf_{Q_I} \omega)^{-1}$ when $p = 1$.

- (c) *There exists a constant $C_3 > 0$ such that for any locally integrable function f and any interval $I \subset \mathbb{R}$,*

$$(9) \quad \left(\frac{1}{|Q_I|} \int_{Q_I} |f(z)| dV(z) \right)^q \mu(Q_I) \leq C_3 \left(\int_{Q_I} |f(z)|^p \omega(z) dV(z) \right)^{q/p}.$$

A special case of Theorem 1.3 appears when μ is a continuous measure with respect to the Lebesgue measure dV in this sense that $d\mu(z) = \sigma(z)dV(z)$, this provides a weak-type two-weight norm inequality for the maximal function.

To prove the sufficient part in the three theorems above, we will observe that the matter can be reduced to the case of the dyadic maximal function. We then use an idea that comes from real harmonic analysis (see for example [6, 7, 19]) and consists of discretizing integrals using appropriate level sets and in our case, the nice properties of the upper-halves of Carleson boxes when they are supported by dyadic intervals. For the proof of the necessity in Theorem 1.2, let us observe that when comes to estimations with loss for the case of the usual Carleson measures for analytic functions, one needs atomic decomposition of functions in the Bergman spaces to apply a method developed by D. Luecking [15]. We do not see how this can be extended here and instead, we show that one can restrict to the dyadic case, and use boundedness of the maximal functions and a duality argument. We note that a duality argument has been used for the same-type of question for weighted Hardy spaces in [8].

Given two positive quantities A and B , the notation $A \lesssim B$ (resp. $B \lesssim A$) will mean that there is an universal constant $C > 0$ such that $A \leq CB$ (resp. $B \leq CA$).

2. USEFUL OBSERVATIONS AND RESULTS

Given an interval $I \subset \mathbb{R}$, the upper-half of the Carleson box Q_I associated to I is the subset T_I defined by

$$T_I := \{z = x + iy \in \mathbb{C} : x \in I, \text{ and } \frac{|I|}{2} < y < |I|\}.$$

Note that $|Q_I| \simeq |T_I|$. We observe the following weighted inequality.

LEMMA 2.1. *Let $1 < p < \infty$. Assume that ω belongs to the Békollé-Bonami class B_p . Then there is a constant $C > 0$ such that for any interval $I \subset \mathbb{R}$,*

$$\omega(Q_I) \leq C[\omega]_{B_p} \omega(T_I).$$

Proof. Using Hölder's inequality and the definition of Békollé-Bonami weight, we obtain

$$\begin{aligned} \frac{|T_I|^p}{|Q_I|^p} &\leq \frac{1}{|Q_I|^p} \left(\int_{T_I} \omega(z) dV(z) \right) \left(\int_{T_I} \omega^{-p'/p}(z) dV(z) \right)^{p/p'} \\ &\leq \frac{1}{|Q_I|^p} \left(\int_{T_I} \omega(z) dV(z) \right) \left(\int_{Q_I} \omega^{-p'/p}(z) dV(z) \right)^{p/p'} \\ &\leq [\omega]_{B_p} \frac{\omega(T_I)}{\omega(Q_I)}. \end{aligned}$$

Thus $\omega(Q_I) \leq [\omega]_{B_p} \left(\frac{|Q_I|}{|T_I|} \right)^p \omega(T_I) \simeq [\omega]_{B_p} \omega(T_I)$. \square

We will also need the following lemma.

LEMMA 2.2. *Let $1 \leq p, q < \infty$ and suppose that ω is a weight, and μ a positive measure on \mathcal{H} . Then the following assertions are equivalent.*

(i) *There exists a constant $C_1 > 0$ such that for any interval $I \subset \mathbb{R}$,*

$$(10) \quad |Q_I|^{-q/p} \left(\frac{1}{|Q_I|} \int_{Q_I} \omega^{1-p'}(z) dV(z) \right)^{q/p'} \mu(Q_I) \leq C_1$$

where $\left(\frac{1}{|Q_I|} \int_{Q_I} \omega^{1-p'}(z) dV(z) \right)^{1/p'}$ is understood as $(\inf_{Q_I} \omega)^{-1}$ when $p = 1$.

(ii) *There exists a constant $C_2 > 0$ such that for any locally integrable function f and any interval $I \subset \mathbb{R}$,*

$$(11) \quad \left(\frac{1}{|Q_I|} \int_{Q_I} |f(z)| dV(z) \right)^q \mu(Q_I) \leq C_2 \left(\int_{Q_I} |f(z)|^p \omega(z) dV(z) \right)^{q/p}.$$

Proof. That (ii) \Rightarrow (i) follows by testing (ii) with $f(z) = \chi_{Q_I}(z) \omega^{1-p'}(z)$ if $p > 1$. For $p = 1$, take $f(z) = \chi_S(z)$ where S a subset of Q_I . One obtains that

$$\frac{\mu(Q_I)}{|Q_I|^q} \leq C_2 \left(\frac{\omega(S)}{|S|} \right)^q.$$

As this happens for any subset S of Q_I , it follows that for any $z \in Q_I$,

$$\frac{\mu(Q_I)}{|Q_I|^q} \leq C_2 (\omega(z))^q$$

which implies (10) for $p = 1$.

Let us check that (i) \Rightarrow (ii). Applying Hölder's inequality (in case $p > 1$) to the right hand side of (11), we obtain

$$\begin{aligned} & \left(\frac{1}{|Q_I|} \int_{Q_I} |f(z)| dV(z) \right)^q \mu(Q_I) \\ & \leq |Q_I|^{-q} \left(\int_{Q_I} \omega^{-p'/p}(z) dV(z) \right)^{q/p'} \mu(Q_I) \left(\int_{Q_I} |f(z)|^p \omega(z) dV(z) \right)^{q/p} \\ & \leq C \left(\int_{Q_I} |f(z)|^p \omega(z) dV(z) \right)^{q/p}. \end{aligned}$$

For $p = 1$, we easily obtain

$$\begin{aligned} \left(\frac{1}{|Q_I|} \int_{Q_I} |f(z)| dV(z) \right)^q \mu(Q_I) & \leq \frac{(\inf_{Q_I} \omega)^{-q}}{|Q_I|^q} \left(\int_{Q_I} |f(z)| \omega(z) dV(z) \right)^q \mu(Q_I) \\ & \leq C \left(\int_{Q_I} |f(z)| \omega(z) dV(z) \right)^q. \end{aligned}$$

The proof is complete. \square

Next, we consider the following system of dyadic grids,

$$\mathcal{D}^\beta := \{2^j ([0, 1) + m + (-1)^j \beta) : m \in \mathbb{Z}, j \in \mathbb{Z}\}, \text{ for } \beta \in \{0, 1/3\}.$$

For more on this system of dyadic grids and its applications, we refer to [1, 11, 12, 13, 14, 18, 20]. When $\beta = 0$, we use the notation $\mathcal{D} = \mathcal{D}^0$ that we call the standard dyadic grid of \mathbb{R} . When I is a dyadic interval, we denote by I^- and I^+ its left half and its right half respectively. We make the following observation which is surely known.

LEMMA 2.3. *Any interval I of \mathbb{R} can be covered by at most two adjacent dyadic intervals I_1 and I_2 in the same dyadic grid such that*

$$|I| < |I_1| = |I_2| \leq 2|I|.$$

Proof. Without loss of generality, we can suppose that $I = [a, b)$. For $x \in \mathbb{R}$, we denote by $[x]$ the unique integer such that $[x] \leq x < [x] + 1$. If $I \in \mathcal{D}$, then there is nothing to say. If $|I| = 1$, then the dyadic interval $[k, k+1)$ where $k = [a]$ covers I .

Let us suppose in general that I is not dyadic. Let j be the unique integer such that

$$(12) \quad 2^{-j} \leq b - a = |I| < 2^{-j+1},$$

and define the set

$$E_{a,b} := \{l \in \mathbb{Z} : a < l2^{-j} \leq b\}.$$

Then $E_{a,b}$ is not empty. To see this, take $k = [a2^j]$, then $(k+1)2^{-j} \leq b$ since if not, we will have $[a, b) \subset [k2^{-j}, (k+1)2^{-j})$ and consequently, $|I| = b - a < [(k+1)2^{-j} - k2^{-j}] = 2^{-j}$ which contradicts (12). Let

$$k_0 := \max\{k : k \in E_{a,b}\}.$$

Then we necessarily have $(k_0 - 2)2^{-j} \leq a$ since if not, $|I| = b - a > k_0 2^{-j} - a > 2^{-j+1}$ and this contradicts (12).

As from the definition of k_0 we have $b \leq (k_0 + 1)2^{-j}$, it comes that if $(k_0 - 1)2^{-j} \leq a$, then the union $[(k_0 - 1)2^{-j}, k_0 2^{-j}) \cup [k_0 2^{-j}, (k_0 + 1)2^{-j})$

covers I , and taking I_1 and I_2 such that $I_1^+ = [(k_0 - 1)2^{-j}, k_0 2^{-j})$ and $I_2^- = [k_0 2^{-j}, (k_0 + 1)2^{-j})$ we get the lemma. If $(k_0 - 1)2^{-j} > a$, then $I \subset I_1 \cup I_2$ where $I_1 = [(l_0 - 1)2^{-j+1}, l_0 2^{-j+1})$, $I_2 = [l_0 2^{-j+1}, (l_0 + 1)2^{-j+1})$ with $k_0 = 2l_0$ if k_0 is even or $k_0 = 2l_0 + 1$ otherwise. The proof is complete. \square

3. PROOF OF THE RESULTS

Let us start with some observations. Recall that given Q_I , its upper-half is the set

$$T_I := \{x + iy \in \mathcal{H} : x \in I, \text{ and } \frac{|I|}{2} < y < |I|\}.$$

It is clear that the family $\{T_I\}_{I \in \mathcal{D}}$ where \mathcal{D} is a dyadic grid in \mathbb{R} provides a tiling of \mathcal{H} .

Next we recall with [18] that given an interval $I \subset \mathbb{R}$, there is a dyadic interval $K \in \mathcal{D}^\beta$ for some $\beta \in \{0, 1/3\}$ such that $I \subseteq K$ and $|K| \leq 6|I|$. It follows in particular that $|Q_K| \leq 36|Q_I|$. Also, proceeding as in the proof of Lemma 2.1 one obtains that $\omega(Q_K) \lesssim [\omega]_{B_p} \omega(Q_I)$. It follows that

$$\frac{1}{\omega(Q_I)} \int_{Q_I} |f(z)| \omega(z) dV(z) \lesssim \frac{1}{\omega(Q_K)} \int_{Q_K} |f(z)| \omega(z) dV(z)$$

and consequently that for any locally integrable function f ,

$$(13) \quad M_\omega f(z) \lesssim \sum_{\beta \in \{0, 1/3\}} M_{d, \omega}^\beta f(z), \quad z \in \mathcal{H}$$

where $M_{d, \omega}^\beta$ is defined as M_ω but with the supremum taken only over dyadic intervals of the dyadic grid \mathcal{D}^β . When $\omega \equiv 1$, we use the notation M_d^β , and if moreover, $\beta = 0$, we just write M_d . In the sequel, we will be proving anything only for the case $\beta = 0$ which is enough and in this case, we write everything without the superscript $\beta = 0$.

3.1. Proof of Theorem 1.1. First suppose that (4) holds and observe that for any interval $I \subset \mathbb{R}$, $1 \leq M_\omega \chi_{Q_I}(z)$ for any $z \in Q_I$. It follows that

$$(\mu(Q_I))^{1/q} \leq \left(\int_{\mathcal{H}} (M_\omega \chi_{Q_I}(z))^q d\mu(z) \right)^{1/q} \leq C_1 \|\chi_{Q_I}\|_{p, \omega} = (\omega(Q_I))^{1/p}$$

which provides that for any interval $I \subset \mathbb{R}$,

$$\mu(Q_I) \leq C_1 (\omega(Q_I))^{q/p}.$$

That is (5) holds.

To prove that (ii) \Rightarrow (i), it is enough by the observations made at the beginning of this section to prove the following.

LEMMA 3.1. *Let $1 < p \leq q < \infty$. Assume that ω is a weight in the class B_p such that (5) holds. Then there is a positive constant C such that for any $f \in L_\omega^p(\mathcal{H})$,*

$$(14) \quad \left(\int_{\mathcal{H}} |M_{d, \omega} f(z)|^q d\mu(z) \right)^{1/q} \leq C_1 \|f\|_{p, \omega}.$$

Proof. Let $a \geq 2$. To each integer k , we associate the set

$$\Omega_k := \{z \in \mathcal{H} : a^k < M_{d,\omega} f(z) \leq a^{k+1}\}.$$

We observe that $\Omega_k \subset \cup_{j=1}^{\infty} Q_{I_{k,j}}$, where $Q_{I_{k,j}}$ is a dyadic cube maximal (with respect to the inclusion) such that

$$\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)dV(z) > a^k.$$

It follows using Lemma 2.1 that

$$\begin{aligned} \int_{\mathcal{H}} (M_{d,\omega} f(z))^q d\mu(z) &= \sum_k \int_{\Omega_k} (M_{d,\omega} f(z))^q d\mu(z) \\ &\leq a^q \sum_k a^{kq} \mu(\Omega_k) \\ &\leq a^q \sum_{k,j} a^{kq} \mu(Q_{I_{k,j}}) \\ &\lesssim a^q \sum_{k,j} \left(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)dV(z) \right)^q \mu(Q_{I_{k,j}}) \\ &\lesssim a^q \sum_{k,j} \left(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)dV(z) \right)^q (\omega(Q_{I_{k,j}}))^{\frac{q}{p}} \\ &\lesssim a^q \left(\sum_{k,j} \left(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)dV(z) \right)^p \omega(Q_{I_{k,j}}) \right)^{\frac{q}{p}} \\ &\lesssim a^q \left(\sum_{k,j} \left(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)dA(z) \right)^p [\omega]_{B_p} \omega(T_{I_{k,j}}) \right)^{\frac{q}{p}} \\ &\lesssim \left(\sum_{k,j} \int_{T_{I_{k,j}}} \left(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)dV(z) \right)^p \omega(w)dV(w) \right)^{q/p} \\ &\lesssim \left(\sum_{k,j} \int_{T_{I_{k,j}}} (M_{d,\omega} f(z))^p dV(z) \right)^{\frac{q}{p}} \\ &\lesssim \left(\int_{\mathcal{H}} |f(z)|^p \omega(z)dV(z) \right)^{\frac{q}{p}}. \end{aligned}$$

The proof of the lemma is complete. \square

3.2. Proof of Theorem 1.2. Let us start by proving the following lemma.

LEMMA 3.2. *Let $1 \leq q < p < \infty$, and let ω be a weight in the class B_p . Assume that μ is a positive measure on \mathcal{H} such that the function K_μ defined by (6) belongs to $L_\omega^s(\mathcal{H})$, $s = \frac{p}{p-q}$. Then there is a constant $C_1 > 0$ such that for any $f \in L_\omega^p(\mathcal{H})$, (14) holds.*

Proof. We proceed as in the proof of Lemma 3.1, using the same notations. Using Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned}
\int_{\mathcal{H}} (M_{d,\omega} f(z))^q d\mu(z) &= \sum_k \int_{\Omega_k} (M_{d,\omega} f(z))^q d\mu(z) \\
&\leq a^q \sum_k a^{kq} \mu(\Omega_k) \\
&\leq a^q \sum_{k,j} a^{kq} \mu(Q_{I_{k,j}}) \\
&\lesssim a^q \sum_{k,j} \left(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)| \omega(z) dV(z) \right)^q \mu(Q_{I_{k,j}}) \\
&= a^q \sum_{k,j} \left(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)| \omega(z) dV(z) \right)^q \frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})} \omega(Q_{I_{k,j}}) \\
&\lesssim A^{q/p} B^{1/s}
\end{aligned}$$

where

$$A = \sum_{k,j} \left(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)| \omega(z) dA(z) \right)^p \omega(Q_{I_{k,j}})$$

and

$$B = \sum_{k,j} \left(\frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})} \right)^s \omega(Q_{I_{k,j}}).$$

From the proof of Lemma 3.1, we already know how to estimate A . Let us estimate B .

$$\begin{aligned}
B &:= \left(\sum_{k,j} \left(\frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})} \right)^s \omega(Q_{I_{k,j}}) \right) \\
&\lesssim \sum_{k,j} \left(\frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})} \right)^s \omega(T_{I_{k,j}}) \\
&\lesssim \sum_{k,j} \int_{T_{I_{k,j}}} \left(\frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})} \right)^s \omega(z) dV(z) \\
&\lesssim \sum_{k,j} \int_{T_{I_{k,j}}} (K_\mu(z))^s \omega(z) dV(z) \\
&\lesssim \int_{\mathcal{H}} (K_\mu(z))^s \omega(z) dV(z) = \|K_\mu\|_{s,\omega}^s.
\end{aligned}$$

The proof of the lemma is complete. \square

We can now prove the theorem

Proof of Theorem 1.2. The proof of the sufficiency follows from Lemma 3.2 and the observations made at the beginning of this section. Let us prove the necessity. For this we do the following observations: first, that condition (4)

implies that there exists a constant $C > 0$ such that for any $f \in L_\omega^p(\mathcal{H})$,

$$(15) \quad \int_{\mathcal{H}} \left(M_{d,\omega}^\beta f(z) \right)^q d\mu(z) \leq C \|f\|_{p,\omega}^q.$$

Second, writing

$$K_{d,\mu}^\beta(z) := \sup_{I \in \mathcal{D}^\beta, z \in Q_I} \frac{\mu(Q_I)}{\omega(Q_I)},$$

it is easy to see that for any $z \in \mathcal{H}$,

$$K_\mu(z) \lesssim \sum_{\beta \in \{0, \frac{1}{3}\}} K_{d,\mu}^\beta(z).$$

Thus to prove that $K_\mu \in L_\omega^s(\mathcal{H})$ if (4) holds, it is enough to prove that (15) implies that $K_{d,\mu}^\beta \in L_\omega^s(\mathcal{H})$. We do this for the standard dyadic grid, i.e for $\beta = 0$.

For $z \in \mathcal{H}$, we write $Q_z = Q_{I_z}$ ($I_z \in \mathcal{D}$) for the smallest Carleson box containing z , and consider the following weighted box kernel

$$K_{d,\omega}(z_0, z) := \frac{1}{\omega(Q_{z_0})} \chi_{Q_{z_0}}(z).$$

For f a locally integrable function, we define

$$K_{d,\omega} f(z_0) = \int_{\mathcal{H}} K_{d,\omega}(z_0, z) f(z) \omega(z) dV(z) = \frac{1}{\omega(Q_{z_0})} \int_{Q_{z_0}} f(z) \omega(z) dV(z).$$

Finally, we define a function g on \mathcal{H} by

$$g(z) := \int_{\mathcal{H}} K_{d,\omega}(\xi, z) d\mu(\xi) = \int_{\mathcal{H}} \frac{\chi_{Q_\xi}(z)}{\omega(Q_\xi)} d\mu(\xi).$$

For any (dyadic) Carleson box Q_I , $I \in \mathcal{D}$, writing Q for Q_I , we obtain

$$\begin{aligned} \frac{1}{\omega(Q)} \int_Q g(z) \omega(z) dV(z) &= \frac{1}{\omega(Q)} \int_Q \left(\int_{\mathcal{H}} K_{d,\omega}(w, z) d\mu(w) \right) \omega(z) dV(z) \\ &= \int_{\mathcal{H}} \int_{\mathcal{H}} \frac{1}{\omega(Q)} \frac{\chi_{Q_w}(z) \chi_Q(z)}{\omega(Q_w)} \omega(z) dV(z) d\mu(w) \\ &\geq \int_Q \frac{1}{\omega(Q)} \int_{\mathcal{H}} \frac{\chi_{Q_w \cap Q}(z)}{\omega(Q_w)} \omega(z) dV(z) d\mu(w) \\ &\gtrsim \frac{1}{\omega(Q)} \int_Q d\mu(w) = \frac{\mu(Q)}{\omega(Q)} \end{aligned}$$

Thus for any $z \in \mathcal{H}$,

$$M_{d,\omega} g(z) \gtrsim \sup_{I \in \mathcal{D}, z \in Q_I} \frac{\mu(Q_I)}{\omega(Q_I)} := K_{d,\mu}(z).$$

Hence if the function g belongs to $L_\omega^s(\mathcal{H})$, then

$$\|K_{d,\mu}\|_{s,\omega} \lesssim \|M_{d,\omega} g\|_{s,\omega} \lesssim \|g\|_{s,\omega}.$$

To finish the proof, we only need to check that $g \in L_\omega^s(\mathcal{H})$ whenever (15) holds.

Let us start by observing the following inequality between $K_{d,\omega}f$ and $M_{d,\omega}f$. Let z_0 be fixed in \mathcal{H} . For any $\xi \in Q_{z_0}$, we have

$$K_{d,\omega}f(z_0) := \frac{1}{\omega(Q_{z_0})} \int_{Q_{z_0}} f(z)\omega(z)dV(z) \leq M_{d,\omega}f(\xi).$$

Thus

$$(16) \quad |K_{d,\omega}f(z)|^{1/q} \leq M_{d,\omega} \left((M_{d,\omega}f)^{1/q} \right) (z), \text{ for any } z \in \mathcal{H}.$$

Now, for any $f \in L_{\omega}^{p/q}(\mathcal{H})$, using (16), (15) and the boundedness of the maximal function, we obtain

$$\begin{aligned} \left| \int_{\mathcal{H}} g(z)f(z)\omega(z)dV(z) \right| &= \left| \int_{\mathcal{H}} \left(\int_{\mathcal{H}} K_{d,\omega}(\xi, z)d\mu(\xi) \right) f(z)\omega(z)dV(z) \right| \\ &= \left| \int_{\mathcal{H}} \left(\int_{\mathcal{H}} K_{d,\omega}(\xi, z)f(z)\omega(z)dV(z) \right) d\mu(\xi) \right| \\ &= \left| \int_{\mathcal{H}} K_{d,\mu}f(\xi)d\mu(\xi) \right| \\ &\leq \int_{\mathcal{H}} |K_{d,\mu}f(\xi)|d\mu(\xi) \\ &= \int_{\mathcal{H}} (|K_{d,\mu}f(\xi)|^{1/q})^q d\mu(\xi) \\ &\lesssim \int_{\mathcal{H}} \left(M_{d,\omega} \left((M_{d,\omega}f)^{1/q} \right) (\xi) \right)^q d\mu(\xi) \\ &\lesssim \left(\int_{\mathcal{H}} (M_{d,\omega}f(z))^{p/q} \omega(z)dV(z) \right)^{q/p} \\ &\lesssim \left(\int_{\mathcal{H}} |f(z)|^{p/q} \omega(z)dV(z) \right)^{q/p}. \end{aligned}$$

Thus there is a constant $C > 0$ such that

$$\|g\|_{s,\omega} := \sup_{f \in L_{\omega}^{p/q}(\mathcal{H}), \|f\|_{p/q,\omega} \leq 1} \left| \int_{\mathcal{H}} g(z)f(z)\omega(z)dV(z) \right| \leq C.$$

The proof is complete. \square

3.3. Proof of Theorem 1.3. We start by the following lemma which tells us that we will only need to restrict to level sets involving the dyadic maximal function.

LEMMA 3.3. *Let f be a locally integrable function. Then for any $\lambda > 0$,*

$$(17) \quad \{z \in \mathcal{H} : Mf(z) > \lambda\} \subset \{z \in \mathbb{D} : M_d f(z) > \frac{\lambda}{68}\}.$$

Proof. Let us put

$$A := \{z \in \mathcal{H} : Mf(z) > \lambda\}$$

and

$$B := \{z \in \mathcal{H} : M_d f(z) > \frac{\lambda}{68}\}.$$

Recall that there is a family $\{Q_{I_j}\}_{j \in \mathbb{N}_0}$ of maximal (with respect to the inclusion) disjoint dyadic Carleson boxes (i.e $I_j \in \mathcal{D}$) such that

$$\frac{4\lambda}{68} \geq \frac{1}{|Q_{I_j}|} \int_{Q_{I_j}} |f| dV > \frac{\lambda}{68}$$

so that $B = \cup_{j \in \mathbb{N}_0} Q_{I_j}$.

Let $z \in A$ and suppose that $z \notin B$. We know that there is an interval I (not necessarily dyadic) such that $z \in Q_I$ and

$$(18) \quad \frac{1}{|Q_I|} \int_{Q_I} |f| dV > \lambda.$$

Recall with Lemma 2.3 that I can be covered by at most two adjacent dyadic intervals J_1 and J_2 (in this order) such that $|I| < |J_1| = |J_2| \leq 2|I|$ so that $Q_I \subset Q_{J_1} \cup Q_{J_2}$. Of course, z belongs only to one (and only one) of the associated boxes Q_{J_1} and Q_{J_2} . Let us suppose that $z \in Q_{J_1}$. Then necessarily, Q_{J_1} is not contained in B since if so, z would belong to B and this would contradict our hypothesis on z . Thus $Q_{J_1} \cap B = \emptyset$ or $Q_{J_1} \supset Q_{I_j}$ for some j and in both cases, because of the maximality of the I_j s, we deduce that

$$\frac{1}{|Q_{J_1}|} \int_{Q_{J_1}} |f| dV \leq \frac{\lambda}{68}.$$

For the other interval J_2 , we have the following possibilities

$$\begin{cases} J_2 = I_j \text{ for some } j \\ J_2 \subset I_j \text{ for some } j \\ J_2 \supset I_j \text{ for some } j \\ J_2 \cap B = \emptyset. \end{cases}$$

If $J_2 \supset I_j$ for some j or $J_2 \cap B = \emptyset$, then because of the maximality of the I_j s,

$$\frac{1}{|Q_{J_2}|} \int_{Q_{J_2}} |f| dV \leq \frac{\lambda}{68}.$$

If $J_2 = I_j$ for some j , then of course,

$$\frac{1}{|Q_{J_2}|} \int_{Q_{J_2}} |f| dV \leq \frac{4\lambda}{68}.$$

It remains to consider the case where $J_2 \subset I_j$ for some j . If $J_2 \subset I_j$, then we can have

$$\begin{cases} J_2 = I_j^- \\ J_2 \subset I_j^- \\ J_2 \subseteq I_j^+ \end{cases}$$

where I_j^- and I_j^+ denote the left and right halves of I_j respectively. If $J_2 \subset I_j^-$ or $J_2 \subseteq I_j^+$, then $J_1 \cap I_j \neq \emptyset$, and this necessarily implies that $J_1 \subset I_j$. Thus $z \in Q_{J_1} \subset Q_{I_j} \subset B$ which contradicts the hypothesis $z \notin B$. Thus the only possible case is $J_2 = I_j^-$ which leads to the estimate

$$\frac{1}{|Q_{J_2}|} \int_{Q_{J_2}} |f| dV \leq \frac{4}{|Q_{I_j}|} \int_{Q_{I_j}} |f| dV \leq \frac{16\lambda}{68}.$$

Thus from all the above analysis, we obtain

$$\begin{aligned}
\frac{1}{|Q_I|} \int_{Q_I} |f| dV &= \frac{1}{|Q_I|} \left(\int_{Q_I \cap Q_{J_1}} |f| dV + \int_{Q_I \cap Q_{J_2}} |f| dV \right) \\
&\leq \frac{|Q_{J_1}|}{|Q_I|} \left(\frac{1}{|Q_{J_1}|} \int_{Q_{J_1}} |f| dV + \frac{1}{|Q_{J_2}|} \int_{Q_{J_2}} |f| dV \right) \\
&\leq 4 \left(\frac{\lambda}{68} + \frac{16\lambda}{68} \right) = \lambda
\end{aligned}$$

which clearly contradicts (18). The proof is complete. \square

We can now prove Theorem 1.3.

Proof of Theorem 1.3. Let us note that by Lemma 2.2, (b) \Leftrightarrow (c). Let us prove that (a) \Leftrightarrow (b).

Let f be a locally integrable function and I an interval. Fix λ such that $0 < \lambda < \frac{1}{|Q_I|} \int_{Q_I} |f| dV$. Then

$$Q_I \subset \{z \in \mathcal{H} : M(\chi_{Q_I} f) > \lambda\}.$$

It follows from the latter and (7) that

$$\mu(Q_I) \leq \frac{C}{\lambda^q} \left(\int_{Q_I} |f(z)|^p \omega(z) dV(z) \right)^{q/p}.$$

As this happens for all $\lambda > 0$, it follows in particular that

$$\mu(Q_I) \left(\frac{1}{|Q_I|} \int_{Q_I} |f| dV(z) \right)^q \leq C \left(\int_{Q_I} |f(z)|^p \omega(z) dV(z) \right)^{q/p}.$$

Next suppose that (9) holds. We observe with Lemma 3.3 that to obtain (7), we only have to prove the following

$$(19) \quad \mu \left(\left\{ z \in \mathcal{D} : M_d f(z) > \frac{\lambda}{68} \right\} \right) \leq \frac{C}{\lambda^q} \|f\|_{p,\omega}^q.$$

We recall that

$$\{z \in \mathcal{H} : M_d f(z) > \frac{\lambda}{68}\} = \cup_{j \in \mathbb{N}_0} Q_{I_j}$$

where the I_j s are maximal dyadic intervals with respect to the inclusion and such that

$$\frac{1}{|Q_{I_j}|} \int_{Q_{I_j}} |f| dV > \frac{\lambda}{68}.$$

Our hypothesis provides in particular that

$$\mu(Q_{I_j}) \lesssim \left(\frac{|Q_{I_j}|}{\int_{Q_{I_j}} |f| dV} \right)^q \left(\int_{Q_{I_j}} |f|^p \omega dV \right)^{q/p}.$$

Thus

$$\begin{aligned}
\mu\left(\left\{z \in \mathbb{D} : M_d f(z) > \frac{\lambda}{68}\right\}\right) &= \sum_j \mu(Q_{I_j}) \\
&\leq \sum_j \left(\frac{|Q_{I_j}|}{\int_{Q_{I_j}} |f| dV}\right)^q \left(\int_{Q_{I_j}} |f|^p \omega dV\right)^{q/p} \\
&\leq \left(\frac{68}{\lambda}\right)^q \sum_j \left(\int_{Q_{I_j}} |f|^p \omega dV\right)^{q/p} \\
&\leq \left(\frac{68}{\lambda}\right)^q \left(\sum_j \int_{Q_{I_j}} |f|^p \omega dV\right)^{q/p} \\
&\leq \left(\frac{68}{\lambda}\right)^q \left(\int_{\mathcal{H}} |f(z)|^p \omega(z) dV(z)\right)^{q/p} \\
&= \left(\frac{68}{\lambda}\right)^q \|f\|_{p,\omega}^q.
\end{aligned}$$

The proof is complete. \square

Taking $d\mu(z) = \sigma(z)dV(z)$, we obtain the following corollary.

COROLLARY 3.4. *Let $1 \leq p, q < \infty$, and ω, σ two weights on \mathcal{H} . Then the following assertions are equivalent.*

- (a) *There is a constant $C_1 > 0$ such that for any $f \in L_\omega^p(\mathcal{H})$, and any $\lambda > 0$,*

$$(20) \quad \sigma(\{z \in \mathcal{H} : Mf(z) > \lambda\}) \leq \frac{C_1}{\lambda^q} \left(\int_{\mathcal{H}} |f(z)|^p \omega(z) dV(z)\right)^{q/p}$$

- (b) *There is a constant $C_2 > 0$ such that for any interval $I \subset \mathbb{R}$,*

$$\begin{aligned}
(21) \quad &|Q_I|^{1/q-1/p} \left(\frac{1}{|Q_I|} \int_{Q_I} \omega^{1-p'}(z) dV(z)\right)^{1/p'} \left(\frac{1}{|Q_I|} \int_{Q_I} \sigma(z) dV(z)\right)^{1/q} \leq C_1 \\
&\text{where } \left(\frac{1}{|Q_I|} \int_{Q_I} \omega^{1-p'}(z) dV(z)\right)^{1/p'} \text{ is understood as } (\inf_{Q_I} \omega)^{-1} \text{ when } \\
&p = 1.
\end{aligned}$$

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